

Chaotic scattering in the regime of weakly overlapping resonances

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We measure the transmission and reflection amplitudes of microwaves in a resonator coupled to two antennas at room temperature in the regime of weakly overlapping resonances and in a frequency range of 3–16 GHz. Below 10.1 GHz the resonator simulates a chaotic quantum system. The distribution of the elements of the scattering matrix S is not Gaussian. The Fourier coefficients of S are used for a best fit of the autocorrelation function of S to a theoretical expression based on random-matrix theory. We find very good agreement below but not above 10.1 GHz.

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Chaotic quantum scattering occurs when Schrödinger waves are scattered by a system with chaotic classical dynamics. For time-reversal invariant chaotic systems, the spectral fluctuations of the eigenvalues coincide [1] with the predictions of the Gaussian orthogonal ensemble (GOE) of real and symmetric random matrices. The eigenvalues manifest themselves as resonances with average spacing D and average width Γ . The theory of chaotic scattering has been largely developed in the framework of nuclear reaction theory [2]. Predictions of the theory have been thoroughly tested both in the regime of isolated resonances ($\Gamma \ll D$) [3] and in the Ericson regime ($\Gamma \gg D$) [4], especially in the context of nuclear physics [5], but also in several other areas of physics [6]. In contradistinction, we are not aware of any thorough investigation of chaotic scattering in the regime of weakly overlapping resonances that would comprise all complex reflection and transmission elements of the scattering matrix. In this Rapid Communication, we present data in that regime and compare these with theoretical predictions.

We use a microwave cavity made of copper coupled to two antennas and measure the response to an external field as a function of radio frequency f . The microwave cavity has the shape of a tilted stadium billiard [7]; see the inset of Fig. 1. The dynamics of the classical stadium billiard is chaotic. The tilted shape was used in order to avoid bouncing-ball orbits between parallel walls. The height of the cavity is 14.6 mm. For frequencies $f \leq f_{\max} = 10.1$ GHz, only a single vertical mode in the microwave cavity is excited. In that regime, the cavity simulates a two-dimensional chaotic quantum system and is a microwave billiard [8]. The experiment is performed at room temperature, with Ohmic losses at the walls of the cavity. A vector network analyzer couples microwave power in and out of the resonator via either one or both antennas and yields the complex elements $S_{ab}(f)$ of the symmetric scattering matrix, where $a, b = 1, 2$. The range $3 \text{ GHz} \leq f \leq 16 \text{ GHz}$ was covered in steps of $\Delta = 250$ kHz in reflection measurements [yielding $S_{11}(f)$ and $S_{22}(f)$] and of $\Delta = 100$ kHz in transmission measurements [yielding $S_{12}(f)$]. Figure 1 gives examples of the measured transmission and reflection intensities.

Figure 2 shows histograms of the distribution of S -matrix elements in two frequency intervals. The distribution of $\text{Re}\{S_{11}\}$ is strongly peaked near 1, especially for the lower

interval, and obviously not Gaussian. The distributions of $\text{Im}\{S_{11}\}$ and $\text{Re}\{S_{12}\}$ deviate from Gaussians (solid lines). The distributions of the phases (rightmost panels) are peaked.

We use the data to construct the S -matrix autocorrelation functions $C_{ab}(\epsilon) = \overline{S_{ab}(f)S_{ab}^*(f+\epsilon) - |S_{ab}(f)|^2}$ for $a, b = 1, 2$. The overbar denotes an average over a frequency window. Three examples for $C_{ab}(\epsilon)$ are displayed as points in the two upper panels of Fig. 3 (data taken below $f_{\max} = 10.1$ GHz) and in the inset of Fig. 4 (data from above f_{\max} where the cavity does not simulate a two-dimensional microwave billiard). The values of the scattering matrix $S_{ab}(f)$ are seen to be correlated, with a correlation width $\Gamma \approx$ several MHz. With $S_{ab}^{\text{fl}} = S_{ab} - \overline{S_{ab}}$ we have also determined the “elastic en-

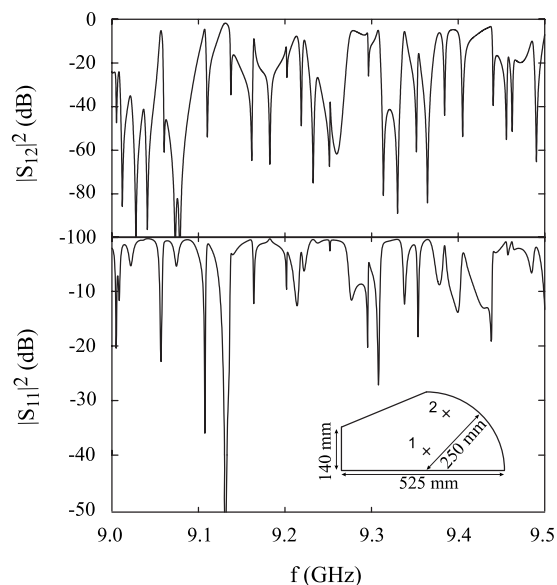


FIG. 1. Absolute squares of the scattering matrix elements S_{ab} for signal transmission from antenna 2 to 1 (upper panel) and reflection at antenna 1 (lower panel) between 9.0 and 9.5 GHz. On the logarithmic decibel scale, $-x$ dB means an attenuation of the microwave power by a factor of $10^{x/10}$. The resonances overlap and create a pattern of fluctuations. Inset: the shape of the two-dimensional microwave resonator used in the experiment. Points 1 and 2 indicate the positions of the antennas.

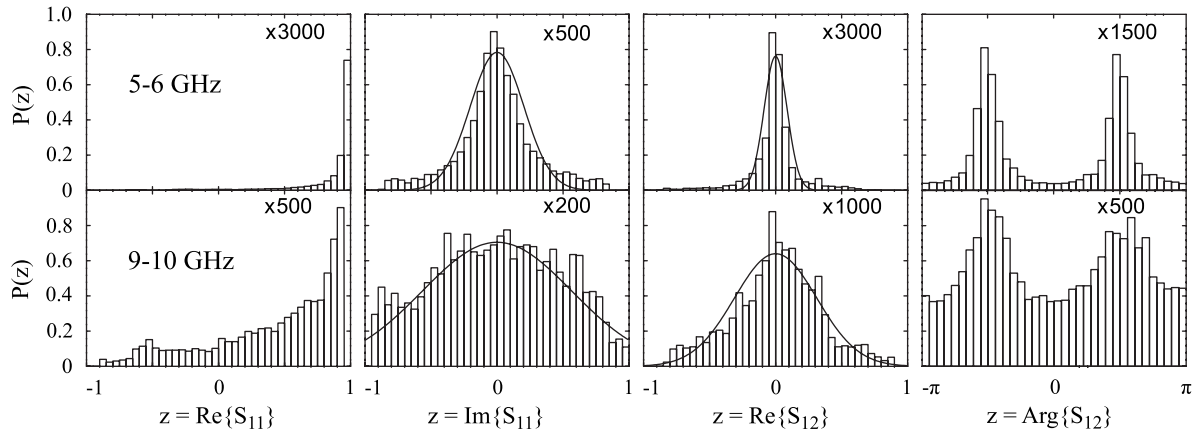


FIG. 2. From left to right: histograms for the scaled distributions of the real and imaginary parts of the reflection amplitude S_{11} and the real part and the phase of the transmission amplitude S_{12} , for the two frequency intervals 5–6 GHz (upper panels) and 9–10 GHz (lower panels). The scaling factors are given in each panel. The solid lines are best fits to Gaussian distributions.

hancement factor” $W = (\overline{|S_{11}^n|^2} \overline{|S_{22}^n|^2})^{1/2} / \overline{|S_{12}^n|^2}$ as a function of f , both from the autocorrelation functions and from the widths of the distributions of the imaginary parts of the scattering matrix (Fig. 2). Both results agree very well and yield

a smooth decrease of W with f from $W \approx 3.5 \pm 0.7$ for $4 \leq f \leq 5$ GHz to $W \approx 2.0 \pm 0.7$ for $9 \leq f \leq 10$ GHz. The computation of the enhancement factors based on a theoretical expression for the S -matrix autocorrelation function introduced below yields the values $W=2.8$ and $W=2.2$, respectively. Moreover, we have converted the scattering functions $S_{ab}(f)$ (measured at M equidistant frequencies with step width Δ) into complex Fourier coefficients $\tilde{S}_{ab}(t)$ with $t \geq 0$. Instead of the Fourier index k we use the discrete time interval $t = k / (M\Delta)$ elapsed after excitation of the resonator. The Fourier coefficient $\tilde{S}_{ab}(0)$ is proportional to $\overline{S_{ab}(f)}$. We find that $\tilde{S}_{12}(0) \approx 0$. Any two complex Fourier coefficients $\tilde{S}_{ab}(t)$ of $S_{ab}(f)$ are uncorrelated random variables [9]. For $t > 0$, the coefficients $\tilde{S}_{ab}(t)$ have an approximately Gaussian distribution about their (t -dependent) mean value [10]. This result is unexpected and was neither predicted theoretically nor found experimentally before.

The Fourier transform $\tilde{C}_{ab}(t)$ of $C_{ab}(\epsilon)$ has Fourier coefficients $x_t = |\tilde{S}_{ab}(t)|^2$. In the lower panels of Fig. 3 (in Fig. 4) we show data for $\log_{10} \tilde{C}_{ab}(t)$ versus t for two values of $\{a, b\}$ (for $\{a, b\} = \{1, 2\}$, respectively). The cutoff at $t = 800$ ns in both figures is due to noise. The $\tilde{S}_{ab}(t)$ being nearly Gaussian, the distribution $P(y_t)$ of $y_t = \ln x_t$ is expected to have approximately the form

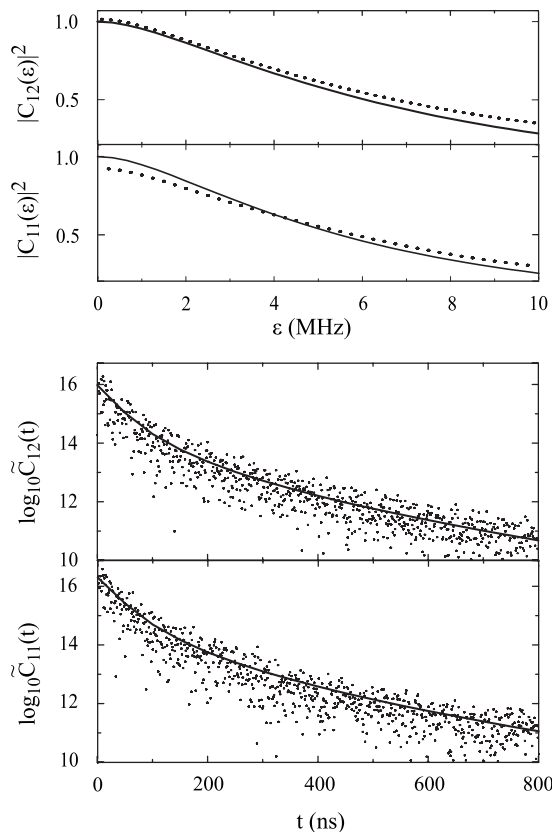


FIG. 3. Upper panels: comparison of the autocorrelation function $C_{ab}(\epsilon)$ constructed from the data (points) and the fit using Eq. (2) (solid line), both normalized by the value of $C(0)$ as given by Eq. (2). Lower panels: Fourier coefficients $\tilde{C}_{ab}(t)$ of the autocorrelation functions (points) and the Fourier transform of the fit of $C_{ab}(\epsilon)$ as given by Eq. (2) to the data (solid line). The elements $S_{12}(f)$ and $S_{11}(f)$ were taken from the frequency window 9–10 GHz.

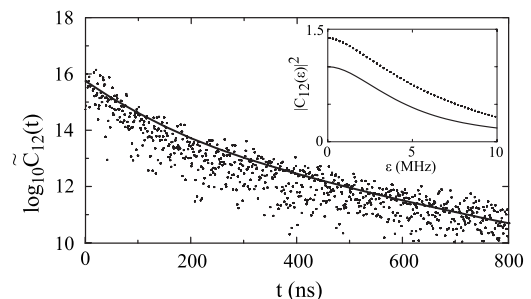


FIG. 4. Same as Fig. 3, but for the scattering function $S_{12}(f)$ taken in the frequency window 12–13 GHz.

$$P(y_t) = \exp(y_t - \eta_t - e^{y_t - \eta_t}), \quad (1)$$

where $\eta_t = \ln \bar{x}_t$ is given by the expectation value of x_t . The maximum of $P(y_t)$ is at $y_t = \eta_t$, and $P(y_t)$ has a strong skewness due to the exponential within the argument of the exponential, in agreement with the experimental data.

In the regime of weakly overlapping resonances, the only theory available is due to Verbaarschot, Weidenmüller, and Zirnbauer (VWZ) [11]. These authors model the scattering matrix S of a time-reversal invariant system in terms of a GOE Hamiltonian matrix of dimension N . In the absence of “direct reactions” [i.e., for $S_{12}(E) = 0$], the relevant parameters of the theory are the “transmission coefficients” $T_c = 1 - |S_{cc}(f)|^2$, which measure the unitarity deficit of the average S matrix. Given the T_c , the theory uses the limit $N \rightarrow \infty$ to predict for all values of Γ/D the S -matrix autocorrelation function

$$C_{ab}(\epsilon) = \frac{1}{8} \int_0^\infty d\lambda_1 d\lambda_2 \int_0^1 d\lambda \mu(\lambda, \lambda_1, \lambda_2) J_{ab}(\lambda, \lambda_1, \lambda_2) \times \exp[-i\pi\epsilon(\lambda_1 + \lambda_2 + 2\lambda)/D] \times \prod_c \frac{(1 - T_c \lambda)}{[(1 + T_c \lambda_1)(1 + T_c \lambda_2)]^{1/2}} \quad (2)$$

in terms of the ratio ϵ/D . To simulate Ohmic absorption by the walls of the cavity, we introduce additional fictitious channels [12] and associated transmission coefficients T_c with $c = 3, 4, \dots$. These are defined below. The product over channels c extends over both the antenna channels and the fictitious channels. The function $J_{ab}(\lambda, \lambda_1, \lambda_2)$ depends on the λ 's and on the transmission coefficients T_a and T_b for the open channels. Both the integration measure $\mu(\lambda, \lambda_1, \lambda_2)$ and J_{ab} are given explicitly in Ref. [11]. The correlation width Γ is actually determined by Eq. (2), but approximately given by the “Weisskopf estimate” $\Gamma \approx [D/(2\pi)] \sum_c T_c$. Equation (2) comprises what is known theoretically in the regime of weakly overlapping resonances. Higher moments of S are not known, not to speak of the complete distribution of S -matrix elements.

Much more is known both for $\Gamma \gg D$ and for $\Gamma \ll D$. In the Ericson regime, the distribution of S -matrix elements is Gaussian; the correlation function $C_{ab}(\epsilon)$ has Lorentzian shape, with Γ given by the Weisskopf estimate [13]; the Fourier transform $\tilde{C}_{ab}(t)$ of $C_{ab}(\epsilon)$ (which describes the decay in time of the modes in the cavity) is exponential in time; for $T_c \approx 1$ (strong absorption) the distribution of the phases of the S_{ab} is constant. For $\Gamma \ll D$, on the other hand, the distribution is far from Gaussian. [Consider, for instance, the single-channel case. The unitarity condition $|S(f)| = 1$ confines $S(f)$ to the unit circle. The phase of $S(f)$ increases by 2π over the width of every resonance and is nearly stationary in between resonances.] The regime $\Gamma \approx D$ interpolates between these two extremes, and we expect a non-Gaussian distribution of $S(f)$. The results in Fig. 2 give experimental information on that distribution and confirm our expectation. With decreasing f , the distributions deviate ever more strongly from Gaussians. As for $\tilde{C}_{ab}(t)$, Eq. (2) predicts a powerlike decay in time, in striking contrast to the exponen-

tial decay valid for $\Gamma \gg D$. That prediction has been discussed and used in Refs. [14] and experimentally tested with microwave resonators in Refs. [12, 15]. However, these papers did not apply any statistical tests based upon a goodness of fit (GOF) as done below.

We model Ohmic absorption by a large number of absorptive channels with very small transmission coefficient each [12]. The product in Eq. (2) over absorptive channels is then replaced by an exponential function of the sum τ_{abs} of the transmission coefficients of these channels, and $C_{ab}(\epsilon)$ depends on $T_1, T_2, \tau_{\text{abs}}$, and D . The Fourier transform was fitted to the x_t data shown in the lower parts of Figs. 3 and 4. We used $T_a = 1 - |S_{aa}|^2$ for $a = 1, 2$ and calculated the mean level spacing D from the Weyl formula [16]. This left τ_{abs} as the only free parameter. In order to allow for secular variations of τ_{abs} , the data taken between 3 and 16 GHz were analyzed in 1-GHz intervals with the help of a maximum-likelihood fit. We find that the sum $T_1 + T_2 + \tau_{\text{abs}}$ increases from 0.11 in the interval 3–4 GHz to 1.15 in the interval 9–10 GHz. The resulting increase of τ_{abs} is consistent with conductance properties of copper. Using the Weisskopf estimate, we find that Γ/D increases from 0.02 to 0.2 over the same range. This shows that we deal with weakly overlapping resonances. The results of the fits are shown as solid lines in the lower two panels of Fig. 3 and in the lower panel of Fig. 4. For an exponential decay in time, the curves in these panels should be straight lines. This is clearly not the case. The solid lines in the upper two panels of Fig. 3 and in the upper panel of Fig. 4 are the Fourier transforms of the VWZ fits. In Fig. 3 they agree well with the data points, up to small discrepancies which are attributed to finite-range-of-data errors. In the upper panel of Fig. 4 the discrepancy between fits and data points is displayed more clearly than in the lower panel.

The quality of the agreement between data and fits in Figs. 3 and 4 is assessed in terms of a highly sensitive GOF test (the Fourier coefficients scatter over more than five orders of magnitude). The fit of Eq. (2) determines the expectation value \bar{x}_t of x_t and, thus, $\eta_t = \ln \bar{x}_t$ in Eq. (1). If the distribution of the y_t were Gaussian, the GOF test would be defined in terms of $\sum_t (y_t - \eta_t)^2$. The appropriate generalization for the distribution $P(y_t)$ in Eq. (1) is the expression $I \propto \sum_t [\exp(y_t - \eta_t) - (y_t - \eta_t) - 1]$; see Chaps. 14 and 16 of Ref. [17]. This quantity is non-negative and vanishes exactly if the data coincide with the model for all t . For large M , I is approximately χ^2 distributed with M degrees of freedom. For each frequency interval of length 1 GHz, we have $M = 2400$, since each of the three excitation functions $S_{11}(f)$, $S_{12}(f)$, and $S_{22}(f)$ contributes 800 Fourier coefficients. We admit a 10% probability for an erroneous decision. The fit using Eq. (2) is accepted in all intervals below $f = 10$ GHz and is rejected in all intervals but one above 10 GHz. A similarly thorough and mathematically reliable test of the theory of chaotic scattering has not been performed before; see Refs. [12, 15]. This fact motivated our work. We conclude that Eq. (2) is compatible with our data as long as the resonator supports only two-dimensional modes and simulates a chaotic billiard. We have numerically simulated the fluctuations above 10.1 GHz under the assumption that the two vertical modes do not interact and the Hamiltonian matrix is block diagonal, each block taken from the GOE. In this way we reproduced quali-

tatively the results of Fig. 4. In the sense that the GOE describes full chaos, a block-diagonal random matrix represents additional symmetries. The disagreement between theory and experiment above 10.1 GHz shows that our test is sensitive to the existence of such symmetries. We conclude that, first, in the regime of overlapping resonances, our test is sensitive to symmetries in a Hamiltonian system and, second, that Eq. (2) is compatible with the data as long as the scattering system is fully chaotic.

We have investigated a chaotic microwave resonator in the regime of weakly overlapping resonances $\Gamma \approx D$. The distributions of S -matrix elements are not Gaussian. In each of 13 frequency intervals we determined 2400 uncorrelated Fourier coefficients of the elements of the scattering matrix. Surprisingly, these have nearly Gaussian distributions. The data were used to test the VWZ theory of chaotic scattering. The predicted nonexponential decay in time of resonator modes and the frequency dependence of the elastic enhancement factor are confirmed. Our GOF test is based on a large number of data points and constitutes the most sensitive test

of the theory of quantum chaotic scattering for weakly overlapping resonances performed so far. We show that the VWZ theory is compatible with the data as long as the resonator simulates a fully chaotic quantum system. The theory can, thus, be used with confidence to predict average cross sections and S -matrix correlation functions. The agreement fails when a second vertical mode appears. This suggests that our analysis may serve as a tool to detect symmetries and/or regular motion within a chaotic system in the regime of overlapping resonances.

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